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Models of the Axiomatic Theory Associated with the Lattice of Subspaces of a Finite-Dimensional Hilbert Space

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It is shown that we can associate with the lattice L of subspaces of a separable Hilbert space an axiomatic theory in sentential logic that reflects some of the basic properties of the simplest types of experimental reports in quantum mechanics. It is also shown that every collection of mutually nonorthogonal elements of L determines a model of the axioms and that, if the Hilbert space is finite dimensional, every model is determined this way.

1. THE PROBLEM

It was shown in Malhas (1987) and later in Malhas (1992) that lattices of propositions isomorphic to L, the lattice of subspaces of a separable Hilbert space, arise quite naturally within classical sentential logic as *the posets of theories*. We shall not be concerned with the concept of "the poset of a theory" in this paper, but with the fact that two *different* theories were proposed, the poset of each of which is isomorphic to L. The theories are different because one is a so-called "theory with orthocomplementation" while the other is not. In each case the theory was obtained as the theory of a set of *valuations* (equivalently, models). In Malhas (1987) the set of models was determined by the set of all pairs of mutually nonorthogonal atoms of L. In Malhas (1992) each model was determined by a *single* atom of L. The two theories, as sets of formulas, have a nonempty intersection containing some nontrivial elements, i.e., formulas which are not tautologies. In what follows we study the theory $\mathcal{T}$ whose axioms are these common nontrivial formulas. In particular we characterize the class of all models of $\mathcal{T}$. It turns

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out that every nonempty collection of pairwise nonorthogonal elements, not necessarily atoms, of L determines a model of the theory and, if L is the lattice of subspaces of a finite-dimensional Hilbert space, then every model arises in this way. A discussion of the physical significance of this fact is beyond the scope of this paper and is currently under preparation.

2. NOTATION AND SETTING

Let L be the lattice of projections on a separable Hilbert space. Quantum mechanics provides "quantization rules" which assign an L-valued measure $\hat{p}$ to each observable p (of the appropriate physical system) (Mackey, 1963 p. 68). For example, the x component of momentum is associated with the L-valued measure which, by the spectral theorem, corresponds to the differential operator $(\hbar/i)\ \partial/\partial x$. If for every Borel set E of the reals we think of the ordered pair (p, E) as a symbol representing the *simple experimental report* [or, as Jauch (1964) called it, a "yes–no experiment"]

The value of the observable p is in E

then the quantization rules alluded to above induce a correspondence between simple experimental reports and elements of L

$$(p, E) \mapsto \hat{p}(E)$$

This mapping is not one–one: We can always find a new observable q and a Borel set G such that $\hat{p}(E) = \hat{q}(G)$. Let Γ be the set of all observables and $\mathbf{B}$ be the set of all Borel subsets of $\mathfrak{R}$. Then the set of all simple experimental reports is $U = \Gamma \times \mathbf{B}$. We take U to be the set of all *sentence symbols* (Chang and Keisler, 1973), or, as we shall also call them, *initial formulas* of a sentential language. The "compound" formulas of the language are obtained from the initial formulas by applying logical connectives $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, . . . to the initial formulas in the reasonable and well-known way. Let W be the set of all, initial and compound, formulas. Of course we must not confuse $(p, E \cup G)$ with $(p, E) \vee (p, G)$. The first is an initial formula, whereas the second is a compound formula. Similarly, we must not confuse (p, E') with $\neg(p, E)$ or $(p, E \cap G)$ with $(p, E) \wedge (p, G)$.

A *model* is simply a subset of U (Chang and Keisler, 1973). If $\alpha \in W$ and M is a model, then we write $M \models \alpha$ to express the idea that "α is *true* in M" or that "M is a model of α" and we write $M \not\models \alpha$ to indicate that M is *not* a model of α. The recursive definition of the symbol $\models$ is as follows:

- For all $\alpha \in W$, if $\alpha \in U$, then $M \models \alpha$ iff $\alpha \in M$.
- For all $\alpha \in W$, $M \models \neg\alpha$ iff $M \not\models \alpha$.
- For all $\alpha, \beta \in W$, $M \models (\alpha \Rightarrow \beta)$ iff $M \models \beta$ whenever $M \models \alpha$.

The meaning of $M \models \alpha \diamond \beta$, where $\diamond$ is any binary connective, can easily be worked out from the above recursive definition, because any $\diamond$ can be defined in terms of $\neg$, $\Rightarrow$. For example, $\alpha \vee \beta$ means $\neg\alpha \Rightarrow \beta$. If $\mathcal{S} \subseteq W$, we write $M \models \mathcal{S}$ iff every member of $\mathcal{S}$ is true in M and say that M is a model of $\mathcal{S}$. If every model of $\mathcal{S}$ is a model of $\alpha \in W$, then we say that α is a *logical consequence* of $\mathcal{S}$. A *closed theory* is a subset $\mathcal{T}$ of W such that if $\alpha \in W$ is a logical consequence of $\mathcal{T}$, then $\alpha \in \mathcal{T}$. In what follows we shall freely use the fact that sentential logic is both sound and adequate, i.e., that the formula α is a logical consequence of a set $\mathcal{S}$ of assumption formulas iff α is *derivable* from $\mathcal{S}$.

3. AXIOMS

We shall use the quantization rules to set up a consistent axiomatic theory in the language of sentential logic. This theory is a description, in axiomatic form, of some basic elements of the observational foundations of quantum mechanics, i.e., a description of the logical structure of the set of yes–no experiments. A formula γ is an *axiom* of the theory iff it satisfies one of the following conditions:

a1. $\gamma = (p, E)$, where $p \in \Gamma$ and $E \in \mathbf{B}$ and $\hat{p}(E) = 1$ the identity position.

a2. $\gamma = ((p, E) \Rightarrow (q, G))$, where $p, q \in \Gamma$ and $E, G \in \mathbf{B}$ and $\hat{p}(E) \leq \hat{q}(G)$

a3. $\gamma = ((p, E) \Rightarrow \neg(q, G))$, where $p, q \in \Gamma$ and $E, G \in \mathbf{B}$ and $\hat{p}(E) \perp \hat{q}(G)$ where $\perp$ denotes orthogonality in L.

From here on, the axioms specified by (a1)–(a3) will be referred to simply as *the axioms*. To refer to one of the three types of axioms we shall refer to *the axioms of type* a1 (or a2 or a3).

Theorem 1. For any observable p, $\neg(p, \emptyset)$ is derivable from the axioms.

Proof. $\hat{p}(\emptyset) \perp \hat{p}(\mathfrak{R})$. Thus $(p, \mathfrak{R}) \Rightarrow \neg(p, \emptyset)$ is an axiom. But $(p, \mathfrak{R})$ is an axiom, by a1. By *Modus Ponens* $\neg(p, \emptyset)$ is derivable. ∎

Let $\mathcal{T}$ be the set of all formulas derivable from the axioms. Then $\mathcal{T}$ is a closed theory. We call $\mathcal{T}$ *the theory associated with L.*

4. MODELS

For all $x, y \in L$, if x is *not* orthogonal to y we write $x \perp' y$. A nonempty set $\mathbf{C}$ of elements of L shall be called a *cluster* iff $\forall x, y \in \mathbf{C}$, $x \perp' y$. Note that no cluster contains the 0 of L, since 0 is orthogonal to every element. Each cluster $\mathbf{C}$ determines a set $M_{\mathbf{C}} \subseteq U$ given by

$$M_{\mathbf{C}} = \{(p, E) \in U: x \leq \hat{p}(E) \text{ for some } x \in \mathbf{C}\} \tag{1}$$

Theorem 2. For every cluster $\mathbf{C}$, $M_{\mathbf{C}}$ is a model of the axioms.

Proof. Let $\mathbf{C}$ be a cluster. Let x be any member of $\mathbf{C}$ and suppose γ is an axiom of type a1. Then $\gamma = (p, E)$ and $\hat{p}(E) = 1$. Since $\hat{p}(E) = 1$, we have $x \leq \hat{p}(E)$. Thus $\gamma \in M_{\mathbf{C}}$.

Now suppose that γ is of the type a2. Then $\gamma = (\alpha \Rightarrow \beta)$, where $\alpha = (p, E)$, $\beta = (q, G)$, and $\hat{p}(E) \leq \hat{q}(G)$. Hence for every $x \in \mathbf{C}$, if $x \leq \hat{p}(E)$, then $x \leq \hat{q}(G)$. This implies that $M_{\mathbf{C}} \models \beta$ whenever $M_{\mathbf{C}} \models \alpha$. Thus $M_{\mathbf{C}} \models (\alpha \Rightarrow \beta)$.

Finally, suppose γ is of the type a3. Then $\gamma = (\alpha \Rightarrow \neg\beta)$, where $\alpha = (p, E)$ and $\beta = (q, G)$ and $\hat{p}(E) \perp \hat{q}(G)$. Then there do not exist elements $x, y \in \mathbf{C}$ such that $x \leq \hat{p}(E)$ and $y \leq \hat{q}(G)$, because the elements of $\mathbf{C}$ are mutually nonorthogonal, whereas $\hat{p}(E)$ is orthogonal to $\hat{q}(G)$. Thus if $M_{\mathbf{C}} \models (p, E)$, then $M_{\mathbf{C}} \not\models (q, E)$. Therefore $M_{\mathbf{C}} \models (\alpha \Rightarrow \neg \beta)$. ■

We see that every cluster determines a model of the axioms. We also see that our axioms are consistent because clusters and, hence, models of the axioms exist.

5. MAIN RESULTS

Does every model of the axioms arise from a cluster as above? We can only give a partial answer here. The answer is "yes" if, *as we shall now assume,* L is the lattice of projections on a *finite*-dimensional Hilbert space. Let M be a model of the axioms.

Lemma 1. For all initial formulas (p, E), if $\hat{p}(E) = 0$, then $(p, E) \notin M$.

Proof. If $\hat{p}(E) = 0$, then $\hat{p}(E) \perp \hat{p}(\mathfrak{R})$. Thus, by a3, $(p, \mathfrak{R}) \Rightarrow \neg(p, E)$ is an axiom. But by a1, $(p, \mathfrak{R})$ is an axiom. Then, by *Modus Ponens*, $\neg(p, E)$ is derivable. Thus $M \models \neg(p, E)$. By the definition of $\models$, $(p, E) \notin M$. ■

An initial formula $(p, E) \in M$ is said to be *minimal in M* if for every $(q, G) \in M$, $\hat{q}(G) \leq \hat{p}(E)$ implies $\hat{q}(G) = \hat{p}(E)$. Let X_M be the set of all $x \in L$ such that $x = \hat{p}(E)$ and $(p, E) \in M$.

Lemma 2. Minimal formulas in M exist.

Proof. Every chain in X_M has finite length because $\mathcal{H}$ is finite dimensional. Every chain in X_M is contained in a chain, in X_M, of maximal length. Select a chain, in X_M, of maximal length and let y be its minimal element.

Then $y = \hat{q}(G)$ for some $(q, G) \in M$ and, by Lemma 1, $y \neq 0$. Clearly (q, G) is minimal for M. ■

Lemma 3. For all $p, q \in \Gamma$ and $E, G \in \mathbf{B}$, if (p, E) and (q, G) are minimal formulas in M, then $\hat{p}(E) \perp' \hat{q}(G)$.

Proof. Suppose to the contrary that $\hat{p}(E) \perp \hat{q}(G)$. By a3, $(p, E) \Rightarrow \neg(q, G)$ is an axiom. Thus $M \models ((p, E) \Rightarrow \neg(q, G))$. This implies that if $(p, E) \in M$, then $(q, G) \notin M$, contradicting the hypothesis that (p, E) and (q, G) are minimal formulas in M and, therefore, are both in M. ■

Let $D = \{x \in L: x = \hat{p}(E)$, where (p, E) is a minimal element in $M\}$.

Lemma 4. The set $\mathbf{D}$ is a cluster.

Proof. The elements of $\mathbf{D}$ are, by Lemma 3, pairwise nonorthogonal. Thus $\mathbf{D}$ is a cluster. ■

Since $\mathbf{D}$ is a cluster, it determines a model of the axioms. This model is obtained from equation (1) by replacing $\mathbf{C}$ by $\mathbf{D}$ and using Theorem 2. Thus the model of the axioms determined by $\mathbf{D}$ is $M_{\mathbf{D}} = \{(p, E) \in U: x \leq \hat{p}(E)$ for some $x \in \mathbf{D}\}$.

Lemma 5. $M \subseteq M_{\mathbf{D}}$.

Proof. Suppose $(q, G) \in M$. If (q, G) is minimal, then (q, G) is in $M_{\mathbf{D}}$. If (q, G) is not minimal, then there is a minimal formula (r, F) in M such that $\hat{r}(F) \in \mathbf{D}$ and $\hat{r}(F) \leq \hat{q}(G)$ and we have $(q, G) \in M_{\mathbf{D}}$. ■

Lemma 6. $M_{\mathbf{D}} \subseteq M$.

Proof. Suppose $(q, G) \in M_{\mathbf{D}}$. Then, from the definitions of $M_{\mathbf{D}}$ and $\mathbf{D}$, $\hat{r}(F) \leq \hat{q}(G)$ for some minimal formula (r, F). Thus $(r, F) \in M$ and $\hat{r}(F) \leq \hat{q}(G)$. By a1, $(r, F) \Rightarrow (q, G)$ is an axiom. Since M is a model, we have $M \models ((r, F) \Rightarrow (q, G))$. From the definition of $\models$ we have that $M \models (q, G)$ whenever $M \models (r, F)$. From the same definition, again, we have $(q, G) \in M$ if $(r, F) \in M$. But we have already deduced that $(r, F) \in M$. Thus $(q, G) \in M$. ■

The next theorem is our main result.

Theorem 3. $M_{\mathbf{D}} = M$.

Proof. By Lemma 5 and Lemma 6. ■

We have seen that every cluster determines a model of the axioms and that every model of the axioms arises from a cluster. Clusters provide all the models of the axioms.

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